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HOPF EXTENSIONS OF ALGEBRAS AND MASCHKE TYPE THEOREMS

BY
YUKIO DOI
Fukui University, Fukui 910, Japan

ABSTRACT
Let B/C be an A-extension for a Hopf algebra A. We consider two Maschke-type questions: first, for an exact sequence of (A, B)-Hopf modules which splits B-linearly, when does it split (A, B)-linearly? Second, for an exact sequence of B-modules which splits C-linearly, when does it split B-linearly? Finally, we consider when the pair of the restriction functor from mod-B to mod-C and the induction functor $(\ )\otimes_C B$ is an adjoint pair of functors.

We shall discuss two different types of Maschke theorems for Hopf algebras, with some applications. Let A be a Hopf algebra over a commutative ring R and B/C an A-extension. In §1 we first show that for every exact sequence of (A, B)-Hopf modules which splits B-linearly, it also splits (A, B)-linearly if there exists a total integral satisfying some specific conditions. As an application in the case when R is a field, we get that an (A, B)-Hopf module is finitely generated projective as a B-module if and only if it is a Hopf module direct summand of $W\otimes B$ for some finite dimensional A-comodule W. We next prove that if A is R-flat, there exists a total integral ϕ with $\phi(A)\subset Z(B)$, and the map $\beta: B\otimes_C B\to B\otimes A$, $b'\otimes b\to\Sigma\, b'b_0\otimes b_1$, is surjective, then β is bijective, that is, B/C is A-Galois.

In §2 we assume the Hopf algebra A is finitely generated projective over R. We first extend "Maschke's Theorem" for group crossed products [Pas, Lemma 1.1] to the case of Hopf Galois extensions. In particular, if there exists a right integral Λ in A with $\varepsilon(\Lambda)=1$, then every A-Galois extension is a separable extension. Finally we prove that if the dual Hopf algebra A^* has a

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nonzero left integral which is unique up to a scalar multiple, as is the case when $\mathrm{Pic}(R)=0$, and B/C is an A-Galois extension, then the pair of the restriction functor from mod-B to mod-C and the induced functor $(-)\otimes_C B$ is an adjoint pair of functors.

We freely use the notations of [S] and [D1].

§1. Throughout, A denotes a Hopf algebra over a commutative ring R with comultiplication Δ, counit ε, and antipode S. We denote by B/C an A-extension B of an R-algebra C; thus B is a right A-comodule algebra with $C=\{b\in B \mid \rho_B(b)=b\otimes 1\}$. Let $\mathcal{M}_B^A$ denote the category of right (A,B)-Hopf modules; thus an object is a right B-module M which is also a right A-comodule such that $\rho_M(mb)=\Sigma\, m_0b_0\otimes m_1b_1$ for $m\in M$ and $b\in B$, and a morphism is a B-module map which is also an A-comodule map.

We now assume that there exists a total integral $\phi: A\to B$, i.e., $\rho_B\phi=(\phi\otimes\mathrm{id})\Delta$ and $\phi(1)=1$. For $M\in\mathcal{M}_B^A$, we define the trace map associated with ϕ as follows;

$$\mathrm{tr}_M: M\to M, \qquad m\to \Sigma\, m_0\phi(S(m_1)).$$

We have that $\mathrm{tr}_M(m)$ lies in $M_0=\{m\in M \mid \rho_M(m)=m\otimes 1\}$ since

$$\rho_M(\mathrm{tr}_M(m))=\Sigma\, m_0\phi(S(m_3))\otimes m_1S(m_2)=\Sigma\, m_0\phi(S(m_2))\otimes\varepsilon(m_1)1$$
$$=\mathrm{tr}_M(m)\otimes 1.$$

Furthermore, the condition $\phi(1)=1$ implies that tr_M is the identity on M_0. In particular, $\mathrm{tr}_B: B\to C$ is a left C-module projection, hence C is a left C-module direct summand of B. Note that M_0 is a right C-module and if $\phi(A)\subset B^c$, the centralizer of C in B, then tr_M is a right C-module map from M onto M_0.

Let $M\in\mathcal{M}_B^A$ and $V\subset M_0$ be a C-submodule of M_0. For $m\in VB\cap M_0$, we have $m=\mathrm{tr}_M(m)\in\mathrm{tr}_M(VB)\subset VC\subset V$. It follows that $VB\cap M_0=V$. Hence, if M is artinian (or noetherian) as a B-module then it is artinian (or noetherian) as a C-module.

For every $M\in\mathcal{M}_B^A$, we can view $M\otimes A$ as a right (A,B)-Hopf module by $(m\otimes a)\cdot b=\Sigma\, mb_0\otimes ab_1$ and $\rho_{M\otimes A}(m\otimes a)=\Sigma\, m\otimes a_1\otimes a_2$, and then $\rho_M: M\to M\otimes A$ becomes a morphism of $\mathcal{M}_B^A$. We now define

$$q_M: M\otimes A\to M, \qquad m\otimes a\to \Sigma\, m_0\phi(S(m_1S(a))).$$

LEMMA. (1) $q_M \rho_M = \mathrm{id}_M$.

(2) *If* $\phi(A) \subset Z(B)$, *the center of* B, *then* q_M *is a* B-*module map, i.e.*, $q_M(m \otimes a)b = \Sigma\, q_M(mb_0 \otimes ab_1)$. *Furthermore, the following diagram is commutative*:

$$\begin{array}{ccc} M \otimes_C B & \xrightarrow{\ \beta_M\ } & M \otimes A \\ {\scriptstyle \mathrm{tr}_M \otimes_C \mathrm{id}}\downarrow & & \downarrow{\scriptstyle q_M} \\ M_0 \otimes_C B & \xrightarrow{\ \Psi_M\ } & M \end{array}$$

where we define $\beta_M(m \otimes b) = \Sigma\, mb_0 \otimes b_1$ *and* $\Psi_M(m \otimes b) = mb$.

(3) q_M *is a morphism of* $\mathcal{M}_B^A$ *if either of the following two conditions is fulfilled*:

(i) A *is involutory* (*i.e.*, $S^2 = \mathrm{id}$), $\phi(A) \subset Z(B)$ *and*

$$\phi(aa') = \phi(a'a) \qquad \text{for all } a, a' \in A,$$

(ii) B *is faithful as an* R-*module and* $\phi(A) \subset R$.

PROOF. (1) Let $m \in M$. Then

$$q_M(\rho_M(m)) = \Sigma\, m_0\phi(S(m_1S(m_2))) = \Sigma\, m_0\phi(S(\varepsilon(m_1)1)) = m\phi(S(1)) = m.$$

(2) We have

$$\begin{aligned} q_M(\Sigma\, mb_0 \otimes ab_1) &= \Sigma\, m_0b_0\phi(S(m_1b_1S(ab_2))) \\ &= \Sigma\, m_0b_0\phi(S(m_1b_1S(b_2)S(a))) \\ &= \Sigma\, m_0b\,\phi(S(m_1S(a))) \\ &= q_M(m \otimes a)b \quad (\text{by } \phi(A) \subset Z(B)), \end{aligned}$$

which shows the first assertion. The diagram clearly commutes from this.

(3) By (2), it suffices to verify that q_M is an A-comodule map.

If (i) holds, then $q_M(m \otimes a) = \Sigma\, m_0\phi(S(m_1)a)$, and hence

$$\begin{aligned} \rho_M q_M(m \otimes a) = \Sigma\, m_0\phi(S(m_3)a_1) \otimes m_1S(m_2)a_2 &= \Sigma\, m_0\phi(S(m_1)a_1) \otimes a_2 \\ &= \Sigma\, q_M(m \otimes a_1) \otimes a_2. \end{aligned}$$

Next suppose that (ii) holds. In this case, we have for $a \in A$, $\Sigma\, \phi(a_1)a_2 = \phi(a)1$, and so $\Sigma\, \phi(S(a_2))S(a_1) = \phi(S(a))1$. If we let $\lambda = \phi S$, then we have

$$\Sigma a_1\lambda(a_2) = \Sigma a_1\phi(S(a_2)) = \Sigma a_1 S(a_2)\phi(S(a_3)) = \Sigma \varepsilon(a_1)\phi(S(a_2))1$$
$$= \phi(S(a))1 = \lambda(a)1.$$

Thus we get for all $a \in A$,

$$\Sigma a_1\lambda(a_2) = \lambda(a)1.$$

Using this we show that q_M is an A-comodule map. Indeed,

$$(q_M \otimes \mathrm{id})\rho_{M\otimes A}(m \otimes a) = \Sigma m_0\lambda(m_1S(a_1)) \otimes a_2 = \Sigma m_0 \otimes \lambda(m_1S(a_1))a_2$$
$$= \Sigma m_0 \otimes m_1S(a_2)\lambda(m_2S(a_1))a_3 = \Sigma m_0 \otimes m_1\lambda(m_2S(a))$$
$$= \Sigma m_0\lambda(m_2S(a)) \otimes m_1 = \rho_M q_M(m \otimes a). \qquad \square$$

We next remove the commutativity hypothesis from [D1, Theorem (1.6)] obtaining a Maschke type theorem for Hopf modules.

THEOREM 1. *Let B/C be an A-extension with a total integral ϕ, and suppose that* (i) *or* (ii) *above holds. Then for every exact sequence of (A, B)-Hopf modules*

$$0 \to M \to N \to P \to 0$$

which splits B-linearly, it also splits (A, B)-linearly.

PROOF. We will show that if $j: M \to N$ is a morphism of $\mathcal{M}_B^A$ such that there exists a B-module map $p: N \to M$ with $pj = \mathrm{id}_M$, then there exists a morphism $\tilde{p}: N \to M$ in $\mathcal{M}_B^A$ such that $\tilde{p}j = \mathrm{id}_M$. Define $\tilde{p}: N \to M$ by $\tilde{p} = q_M(p \otimes \mathrm{id})\rho_N$. Then for $n \in N$ and $b \in B$,

$$\tilde{p}(nb) = q_M(\Sigma p(n_0)b_0 \otimes n_1b_1) \qquad \text{(since } p \text{ is } B\text{-linear)}$$
$$= q_M(\Sigma p(n_0) \otimes n_1)b \qquad \text{(by (2) of Lemma)}$$
$$= \tilde{p}(n)b, \qquad \text{thus } \tilde{p} \text{ is a } B\text{-module map.}$$

Moreover we have

$$\rho_M\tilde{p} = \rho_M q_M(p \otimes \mathrm{id})\rho_N$$
$$= (q_M \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)(p \otimes \mathrm{id})\rho_N \qquad \text{(since } q_M \text{ is an } A\text{-comodule map)}$$
$$= (q_M \otimes \mathrm{id})(p \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)\rho_N$$

$$= (q_M \otimes \mathrm{id})(p \otimes \mathrm{id} \otimes \mathrm{id})(\rho_N \otimes \mathrm{id})\rho_N$$

$$= (\tilde{p} \otimes \mathrm{id})\rho_N,$$

this shows that $\tilde{p}$ is an A-comodule map.

Finally, $\tilde{p}j = q_M(p \otimes \mathrm{id})\rho_N j = q_M(p \otimes \mathrm{id})(j \otimes \mathrm{id})\rho_M = q_M\rho_M = \mathrm{id}_M$ (by (1) of Lemma). □

We do not know whether Theorem 1 holds for the more general hypothesis $\phi(A) \subset Z(B)$.

As an application of Theorem 1, we obtain a characterization that an (A, B)-Hopf module be projective as a B-module:

THEOREM 2. *Assume R is a field. Let B/C be an A-extension with a total integral ϕ, and suppose that the condition* (i) *or* (ii) *holds. Then an (A, B)-Hopf module P is projective as a B-module if and only if P is an (A, B)-Hopf module direct summand of $W \otimes B$ for some A-comodule W, where $W \otimes B$ is considered as (A, B)-Hopf module via $(w \otimes b)b' = w \otimes bb'$ and $\rho_{w\otimes B}(w \otimes b) = \Sigma\, w_0 \otimes b_0 \otimes w_1 b_1$. Moreover we may assume that the above W is finite dimensional when P is a finitely generated B-module.*

PROOF. The "if" part is obvious since $W \otimes B$ is a free B-module. To prove the "only if" part, let $G \subset P$ be a subset that generates P as a B-module. Let W be the smallest A-subcomodule of P containing G. We know that if G is a finite set then W is finite dimensional. Observe that the map $W \otimes B \to P$, $w \otimes b \to wb$, is a surjective morphism of $\mathcal{M}_B^A$. Since P is B-projective, it follows from Theorem 1 that P is an (A, B)-Hopf module direct summand of $W \otimes B$.

REMARK. Theorems 1 and 2 are very close in spirit to [B, Proposition (6.6) and Corollary (6.7)], with the Hopf algebra A replaced by a reductive group acting on an affine scheme $\mathrm{Spec}(B)$.

Recall that an A-extension B/C is said to be A-Galois if the map $\beta : B \otimes_C B \to B \otimes A$, $b' \otimes b \to \Sigma\, b'b_0 \otimes b_1$, is bijective. The next result is an application of (2) in Lemma, and is an improvement of [D1, Theorem (2.5)].

THEOREM 3. *Let A be a Hopf algebra which is a flat R-module. Let B/C be an A-extension such that there exists a total integral $\phi : A \to B$ with $\phi(A) \subset Z(B)$. If the map β is surjective then it is bijective, that is, B/C is an A-Galois extension.*

PROOF. We first note that for any $M \in \mathcal{M}_B^A$, the map $\Psi_M : M_0 \otimes_C B \to M$, $m \otimes b \to mb$, is surjective. This follows from the commutativity of the diagram in (2) of Lemma, since β_M (and q_M) is surjective by $\beta_M = \mathrm{id}_M \otimes_B \beta$.

Let N be the kernel of Ψ_M. Since Ψ_M is a morphism of $\mathcal{M}_B^A$ and A is a flat R-module, N becomes a right (A, B)-Hopf module, and hence $\Psi_N : N_0 \otimes_C B \to N$ is surjective. But we must have $N_0 = 0$ since $(M_0 \otimes_c B)_0 = M_0$ [DT, the proof in (2.11)(b)], and so we obtain $N = 0$. Thus Ψ_M is bijective for any $M \in \mathcal{M}_B^A$. In particular, β is bijective since we may identify $\beta = \Psi_{B \otimes A}$ [D1, (2.3)]. □

EXAMPLE. $A = RG$, a group algebra (G not necessarily finite). It is well known that an A-extension of an algebra C is precisely a G-graded algebra $B = \bigoplus_{g \in G} B_g$ with $B_1 = C$ via $\rho(b) = \Sigma_{g \in G} b_g \otimes g$, and the category $\mathcal{M}_B^A$ is precisely the category of G-graded B-modules. Any R-linear map $\phi : RG \to B$ is determined by the image of G under ϕ, and it is easy to see that ϕ is a total integral if and only if $\phi(g) \in B_g$ for all $g \in G$ and $\phi(1) = 1$. Thus all RG-extensions have total integrals. For instance, if we set ψ by $\psi(g) = \delta_{g,1}$, then this is a total integral. Hence we may use Theorem 1 so that if $N \subset M$ are G-graded B-modules such that N has a B-complement in M, then it has a (G, B)-complement in M, as is well known. Theorem 3 tells us that an RG-Galois extension means strongly G-graded algebra, which is due to Ulbrich [U]. It is easy to see that a total integral $\phi : RG \to B$ is convolution-invertible if and only if $\phi(g) \in B_g \cap U(B)$ for all $g \in G$. Thus an RG-cleft extension of C [cf. D2] is nothing but a group crossed product $C \bigstar G$.

§2. We assume throughout this section that A is a finitely generated projective Hopf algebra over a commutative ring R. An element $\Lambda \in A$ is called a right integral if $\Lambda a = \varepsilon(a)\Lambda$ for all $a \in A$. The Hopf algebra A has a non-zero right integral, as is the case when R is a field.

In [Pas, Lemma 1.1], Maschke's classical theorem on group algebras of finite groups was extended to crossed products $C \bigstar G$. We first extend this to the case of Hopf Galois extensions B/C, with essentially the same proof:

THEOREM 4. *Let $\Lambda \in A$ be a non-zero right integral and B/C an A-Galois extension. Let $V' \subset V$ be right B-modules having no $\varepsilon(\Lambda)$-torsion. If V' is a C-module direct summand of V, then there exists a B-submodule U of V such that $V' \oplus U$ is essential in V as a C-module. Furthermore, if $V = V\varepsilon(\Lambda)$, then V' is a B-module direct summand of V.*

PROOF. Let $\Sigma x_i \otimes y_i \in B \otimes_C B$ be the element such that $1 \otimes \Lambda = \beta(\Sigma x_i \otimes y_i)$. Then we claim that the following formulas hold:

(a) $\Sigma bx_i \otimes y_i = \Sigma x_i \otimes y_i b$ for all $b \in B$,

(b) $\Sigma x_i y_i = \varepsilon(\Lambda)1$.

Indeed, we have $\beta(\Sigma bx_i \otimes y_i) = b \otimes \Lambda$ since β is a left B-module map. On the other hand,

$$\beta(\Sigma x_i \otimes y_i b) = (1 \otimes \Lambda)\rho_B(b) = \Sigma b_0 \otimes \Lambda b_1 = \Sigma b_0 \otimes \varepsilon(b_1)\Lambda = b \otimes \Lambda.$$

Thus (a) follows from the injectivity of β. (b) follows immediately from $(\mathrm{id} \otimes \varepsilon)\beta$.

Thus a standard argument shows that for any B-module V_1, V_2, and any C-module map $p : V_1 \to V_2$, we can form the B-module map;

(c) $\tilde{p} : V_1 \to V_2$, $v \to \Sigma\, p(vx_i)y_i$.

Now let $p : V \to V'$ be a C-module map such that $p(v') = v'$ for all $v' \in V'$. Then $\tilde{p}(v') = \Sigma\, p(v'x_i)y_i = \Sigma\, v'x_i y_i = v'\varepsilon(\Lambda)$. Hence if $U = \mathrm{Ker}(\tilde{p})$ then U is a B-submodule with $U \cap V' = 0$ since V has no $\varepsilon(\Lambda)$-torsion. We also have that $V\varepsilon(\Lambda) \subset U \oplus V'$ since $v\varepsilon(\Lambda) - \tilde{p}(v) \in U$ for all $v \in V$. It follows that $U \oplus V'$ is essential in V as a C-module. Especially if $V = V\varepsilon(\Lambda)$ then $V = U \oplus V'$ and so V' is a B-module direct summand of V. □

REMARK. As a consequence we obtain that every Galois extension B/C is a separable extension for A having a right integral Λ in A with $\varepsilon(\Lambda)^{-1} \in R$. If C is artinian then so is B because B is finitely generated as a left (or right) C-module [KT, Theorem (1.7)]. Consequently if C is semisimple artinian then so is B. This is a generalization of [BM, Theorem 2.6] since Hopf crossed products $C \#_\sigma A$ are A-Galois extensions of C.

More generally, for any finitely generated projective Hopf algebra A, the author and M. Takeuchi have obtained in [DT, Theorem (3.14)] a necessary and sufficient condition for an A-Galois extension B/C to be separable, using the method of Miyashita–Ulbrich actions.

We next consider the induced functor $F = (-) \otimes_C B : \text{mod-}C \to \text{mod-}B$ and the restriction functor $H : \text{mod-}B \to \text{mod-}C$. F is always a left adjoint of H. It is interesting to ask when F is a right adjoint of H [RR, 1.1(B)]. We shall give a positive answer to this question.

We denote by A^* the dual Hopf algebra of A. A^* is a left A^*-module by multiplication, and hence a right A-comodule. Also we regard A^* as a right A-module where $(a^* \leftharpoonup a)(a') = a^*(a'S(a))$ for $a^* \in A^*$ and $a, a' \in A$. Then A^* becomes a right Hopf module over A, and hence $J \otimes A \to A^*$, $\lambda \otimes a \to \lambda \leftharpoonup a$, is an isomorphism as A-Hopf modules [S, Section 4.1; Par, Section 3; KT, (1.1)(ii)], where J denotes the space of all left integrals in A^*. Consequently J is a finitely generated projective rank 1 R-module. Assume that J is a free R-module (e.g., $\mathrm{Pic}(R)=0$). Take $\lambda\ (\neq 0) \in J$. Then $\Theta : A \to A^*$, $a \to \lambda \leftharpoonup a$, is a right A-module and right A-comodule isomorphism. Let $\Lambda \in A$ be the element satisfying $\lambda \leftharpoonup \Lambda = \varepsilon$. Then we have:

LEMMA. (i) *Λ is a right integral with $\lambda(\Lambda) = 1 = \lambda(S(\Lambda))$.*

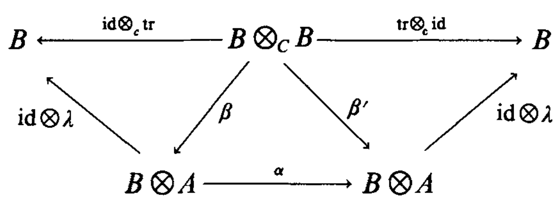
(ii) *Let B/C be an A-Galois extension, and let $\Sigma\, x_i \otimes y_i = \beta^{-1}(1 \otimes \Lambda)$. Also we define* $\mathrm{tr}(b) = \Sigma\, b_0\lambda(b_1)$ *for each $b \in B$. Then* tr *is a C-bimodule map from B onto C, and the following formula holds:*

(d) $\Sigma\, x_i\, \mathrm{tr}(y_i) = 1 = \Sigma\, \mathrm{tr}(x_i) y_i$.

PROOF. (i) Since $\Theta(\Lambda a) = \Theta(\Lambda) \leftharpoonup a = \varepsilon \leftharpoonup a = \varepsilon(a)\varepsilon = \varepsilon(a)\Theta(\Lambda) = \Theta(\varepsilon(a)\Lambda)$ for all $a \in A$, hence Λ is a right integral. Substituting 1 to $\lambda \leftharpoonup \Lambda = \varepsilon$, we have $\lambda(S(\Lambda)) = 1$.

Since $\Theta : A \to A^*$ is a right A-comodule map it is a left A^*-module map, where we regard A as a left A^*-module via $a^* \rightharpoonup a = \Sigma\, a_1 a^*(a_2)$ for $a^* \in A^*$, $a \in A$. It follows that $\Theta^{-1}(a^*) = a^* \rightharpoonup \Lambda$ since $\Theta(a^* \rightharpoonup \Lambda) = a^*\Theta(\Lambda) = a^*\varepsilon = a^*$. In particular, we have $\lambda \rightharpoonup \Lambda = \Theta^{-1}(\lambda) = \Theta^{-1}\Theta(1) = 1$, so $\Sigma\, \Lambda_1 \lambda(\Lambda_2) = 1$. Applying ε to this, we get $\lambda(\Lambda) = 1$. We note that this argument is similar in spirit to [LR, Proposition 1.1].

(ii) By the definition of left integrals in A^*, $a^*\lambda = a^*(1)\lambda$ for all $a^* \in A^*$, equivalently we have $\Sigma\, a_1\lambda(a_2) = \lambda(a)1$ for all $a \in A$. From this it is easy to see that $\mathrm{tr}(b)$ is in C and tr is a C-bimodule map. Furthermore we have the following commutative diagram:



here maps β' and α are defined by $\beta'(b \otimes b') = \Sigma\, b_0 b' \otimes b_1$ and $\alpha(b \otimes a) = \Sigma\, b_0 \otimes b_1 S(a)$. Combining this with $\lambda(\Lambda) = 1 = \lambda(S(\Lambda))$ above, we obtain the formula (d). □

THEOREM 5. *Let A be a finitely generated projective Hopf algebra over R such that the space of all left integrals in A^* is a free R-module. Let B/C be an A-*

Galois extension, and let V be a right B-module and W a right C-module. Then there exists an R-module isomorphism

$$\mathrm{Hom}_C(V, W) \to \mathrm{Hom}_B(V, W \otimes_C B)$$

which is functorial in V and W.

PROOF. For $f \in \mathrm{Hom}_C(V, W)$, we define a B-module map $\xi(f): V \to W \otimes_C B$ by $\xi(f) = (\Phi_W f)^{\sim}$ (see (c) in the proof of Theorem 4), where $\Phi_W: W \to W \otimes_C B$, $w \to w \otimes_C 1$, thus $\xi(f)(v) = \Sigma f(vx_i) \otimes_C y_i$. Conversely, for $g \in \mathrm{Hom}_B(V, W \otimes_C B)$, define $\eta(g) \in \mathrm{Hom}_C(V, W)$ by $\eta(g) = (\mathrm{id} \otimes_C \mathrm{tr})g$. Then we have for all $v \in V$,

$$\eta(\xi(f))(v) = (\mathrm{id} \otimes_C \mathrm{tr})(\Sigma f(vx_i) \otimes y_i) = \Sigma f(vx_i)\mathrm{tr}(y_i) = \Sigma f(vx_i \,\mathrm{tr}(y_i)) = f(v)$$

(by (d)).

Let $v \in V$. If we denote $g(v) = \Sigma w_i \otimes_C b_j$ then for all $b \in B$ we have $g(vb) = \Sigma w_i \otimes_C b_j b$. It follows that

$$\begin{aligned}
\xi(\eta(g))(v) &= \Sigma\, (\mathrm{id} \otimes_C \mathrm{tr}) g(vx_i) \otimes_C y_j \\
&= \sum_{i,j} w_j \,\mathrm{tr}(b_j x_i) \otimes_C y_i \\
&= \sum_{i,j} w_i \otimes_C \mathrm{tr}(b_j x_i) y_i \\
&= \sum_j w_j \otimes_C \left(\sum_i \mathrm{tr}(x_i) y_i \right) b_j \qquad \text{(by (a))} \\
&= g(v) \qquad \text{(by (d))}.
\end{aligned}$$

This shows that $\eta\xi$ and $\xi\eta$ both are identities, so that we have our desired isomorphism, which is clearly functorial in V and W. □

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